Tentamen Numerical Mathematics 2
July 10, 2009

Duration: 3 hours.
In front of the questions one finds the weights used to determine the final mark.

Problem 1

a. [2] Assume A is symmetric. Show that the condition for positive definiteness $(x, Ax) > 0$ if $||x|| \neq 0$ is equivalent to the condition that all eigenvalues of A should be positive.

b. [3] Show by the Gershgorin circle theorems and using (a) that the following matrix admits a Cholesky factorization and make this factorization:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

c. [2] Draw the graph associated to the following matrix and determine from the graph whether the matrix is reducible or not:

$$A = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 0 & 3 & 0 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Problem 2

a. [3] Let A be diagonizable, i.e. there exists a nonsingular P such that $P^{-1}AP = D$ with D a diagonal matrix. Show that for any eigenvalue μ of the perturbed matrix $A + E$ it holds that

$$\min_{\lambda \in \sigma(A)} |\lambda - \mu| \leq K_2(P)||E||_2$$

where $K_2(*)$ is the condition number of its argument in the 2-norm. For which matrices A is $K_2(P) = 1$?

b. [3] Show that the power method converges to the eigenvector corresponding to the eigenvalue with largest magnitude.

c. [2] Describe the QR method to determine eigenvalues of a matrix A. Where does it converge to if the matrix A is real? What determines its convergence? Why is a shift introduced?

Exam questions continue on other side

Problem 3

a. [3] Describe how orthogonal polynomials are constructed from the basic polynomials $\{1, x, x^2, ...\}$. How are in particular the Chebyshev and Legendre polynomials constructed and how are the associated innerproducts defined?

b. [3] How is the minimax problem defined and where is it used for? Why do we turn to the least squares approach to find an approximation to the best approximation of a certain function? And why is the Chebyshev expansion of the function the favourite approximation to the best approximation?

c. [3] Show that the DFT $\hat{f}_k = \frac{1}{N}\sum_0^{N-1} f_j w^{jk}$ for $k = 0, ..., N-1$ where $w = \exp(-i2\pi/N)$ of length $N = 2M$ can be evaluated by two DFTs of length M. What is the advantage of this?

Problem 4

a. [3] Suppose we want to solve the system of ODEs given by $\frac{d}{dt}y = Ay$ with $y(0)$ given. Here, y is a vector and A a matrix. Which linear systems have to be solved if we apply to this the Runge-Kutta methods with the following Butcher arrays

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 0 & 1 \end{array} \qquad \begin{array}{c|cc} 1/2 & 1/2 & 0 \\ 1 & 1/3 & 2/3 \\ \hline & 0 & 1 \end{array} \qquad \begin{array}{c|cc} 1/2 & 1/4 & 1/4 \\ 1 & 1/2 & 1/2 \\ \hline & 0 & 1 \end{array}$$

b. [3] Let $f(x) \in C^1$ be defined on [0,2] and $f(0) = 0$. Show that $||f||_0 \leq c||f'||_0$ where the norm is generated by the inner product $(u, v) = \int_0^2 u(x)v(x)dx$ and give c. What is the relevance of this inequality for one-dimensional Poisson problems on the specified interval?

c. [3] Give the discretisation of $u_t = u_{xx}$ on [0,1] and $t > 0$ using the trapezoidal method in time and the Galerkin approach in space. We have Dirichlet boundary conditions at 0 and 1 and an initial condition for $t = 0$.